Independence of Irrelevant Opinions and Symmetry Imply Liberalism

Wonki Jo Cho* Bung-Ghi Ju†

November 9, 2014

Abstract

There are at least three groups and each person has to be identified as a member of one of these groups. A social decision rule determines the memberships for the groups based on individual opinions on who belong to what groups. Our main axiom is the requirement that the membership for each group, say the group of J's, should depend only on the opinions on who is a J and who is not (that is, independently of the opinions on who is a K or an L). This shares the spirit of Arrow's independence of irrelevant alternatives and therefore is termed *independence of irrelevant opinions*. Our investigation of the multinary group identification and the independence axiom reports a somewhat different message from the celebrated impossibility result by Arrow (1951). In particular, we show that the independence axiom, together with *symmetry* and *non-degeneracy*, implies the liberal rule (each person self-determines her own membership), which gives a theoretical foundation for the self-identification method commonly used for racial or ethnic classifications.

JEL Classification Numbers: C0; D70; D71; D72

Key Words: group identification; independence of irrelevant opinions; symmetry; liberalism; one-vote rules

*School of Social Sciences, University of Manchester, Oxford Road, Manchester, United Kingdom, M13 9PL; jo.cho@manchester.ac.uk

†Department of Economics, Seoul National University, Gwanak-ro 1, Gwanak-gu, Seoul, South Korea, 151-746; bgju@snu.ac.kr

1 Introduction

In countries with diverse demographic characteristics such as China, India, Russia, the United Kingdom, the United States, etc., ethnic or racial classification serves as a basis for evaluating public policies, government bodies, and other social institutions in terms of equal opportunity and anti-discrimination. This classification issue is often complicated by the fact that memberships for a large number of ethnic groups should be determined all at the same time. For instance, the 2011 UK Census recognizes eighteen ethnicity categories[1] and the 2010 China Census fifty-six.[2]

Group identification (Kasher and Rubinstein, 1997) formalizes the problem of classifying individuals. However, the literature largely focuses on the binary case (Samet and Schmeidler, 2003; Sung and Dimitrov, 2005; Dimitrov, Sung and Xu, 2007; Houy, 2007; Miller, 2008; Çengelci and Sanver, 2010; Ju, 2010, 2013). In the binary model, there is only one group, say the group of J's, whose membership is to be determined. Each person has an opinion on who belong to the group. The question is how to aggregate individual opinions and identify members (J's) and non-members (non-J's).

Such a binary model, when applied to multinary problems involving three or more groups, dismisses all opinions on the other groups, say K or L. Person i may view person j not to be in J but to be in K or L; in either case, the identification of group J through the binary model remains the same. Thus, implicit in the binary model is the principle that the identification of the group under question should not be tainted by irrelevant opinions on the other groups, which is reminiscent of Arrow's independence of irrelevant alternatives (Arrow, 1951).[3] We propose this principle as an axiom, termed *independence of irrelevant opinions*, for social decision rules over multinary group identification problems. Despite the wide use of multinary classifications in the aforementioned countries and the celebrity of the independence axiom in Social Choice Theory, as far as we know, there has been no earlier investigation of the multinary group identification, not to speak of the independence axiom therein. Our main objective is to scrutinize

[1] Office for National Statistics (HTTP://WWW.ons.gov.uk/ons/guide-method/measuring-equality/equality/ethnic-nat-identity-religion/ethnic-group/index.html). Retrieved July 25, 2014.

[2] National Bureau of Statistics of the People's Republic of China (HTTP://WWW.stats.gov.cn/tjsj/pcsj/rkpc/6rp/indexch.htm). Retrieved July 25, 2014.

[3] See also Hansson (1969) and Fishburn (1970) for their discussion on the role of the independence axiom in Arrow's impossibility theorem.

independence of irrelevant opinions in the multinary group identification.

In our model, there are three or more groups, and each person needs to be identified as a member of one of the groups. Taking as input individual opinions on who belong to what groups, a (social decision) rule determines memberships for the groups. Our main axiom for rules, *independence of irrelevant opinions*, requires that the membership for each group should be decided based solely on the opinions on who belong to that group and who do not (that is, independently of the opinions on who belong to the other groups). It is a variant of Arrow's independence axiom for preference aggregation rules and is vacuous in the binary group identification, as is Arrow's independence when there are only two alternatives.

We show that *independence of irrelevant opinions*, together with the basic condition of *non-degeneracy* (there should be no person who is always put in one fixed group, regardless of opinions), implies a simple method of identifying each person using only "one vote" (Theorem 1). We call these rules the one-vote rules, noting the connection with the one-vote rules in the binary model (Miller, 2008). For example, a dictatorial rule determines each person's membership following the dictator's opinion; each person is a J when, and only when, the dictator believes so. Another example is the liberal rule, according to which everyone self-determines her membership. There are many other one-vote rules as well. However, when *symmetry* (the names of persons should not matter; Samet and Schmeidler, 2003) is added, the liberal rule is the unique rule satisfying the three axioms (Theorem 2). Therefore, our investigation of multinary group identification and the independence axiom therein reports a somewhat different message from the well-known impossibility result in preference aggregation theory by Arrow (1951) and Blau (1957).[4]

The liberal rule, or the self-identification method, is the most common way of identifying one's ethnicity and race. For example, the 2011 UK Census uses this method and the Office for National Statistics of the UK government explains the reason as follows:

> Membership of an ethnic group is something that is subjectively meaningful to the person concerned, and this is the principal basis for ethnic categorization in the United Kingdom. So, in ethnic group questions, we are unable to base ethnic

[4]Blau (1972) provides a simpler proof of Arrow's theorem when there are at least five alternatives.

identification upon objective, quantifiable information as we would, say, for age or gender. And this means that we should rather ask people which group they see themselves as belonging to.[5]

The subjective nature of the classification is the main reason for using the census where all persons concerned can report their opinions. Nevertheless, the Office does not provide a more fundamental basis for the self-identification method or the principles underlying it. Our characterization of the liberal rule by *independence of irrelevant opinions*, *symmetry*, and *non-degeneracy* reveals what those principles can be and serves as a formal justification.

Our results rest chiefly on *independence of irrelevant opinions* and the assumption that there are three or more groups. In the binary model, *independence of irrelevant opinions* has no bite, and there are numerous rules other than the liberal rule satisfying both *symmetry* and *non-degeneracy.* The consent rules by Samet and Schmeidler (2003) are examples. The liberal rule is a special case in this family, with the minimum consent quotas. Depending on the choice of consent quotas, a consent rule can also be "democratic" in that everyone's vote counts equally; e.g., the majority rule with the consent quota $\frac{n+1}{2}$, where n is the number of persons.

One important reason why *independence of irrelevant opinions* turns out so strong is that social decisions in our model identify each person as a member of exactly one group; thus, decisions partition the set of persons into groups (no one belongs to none of the groups or to two or more groups at the same time).[6] When the partitioning property in our model is relaxed slightly, our results no longer hold and the family of rules satisfying *independence of irrelevant opinions* and *non-degeneracy* becomes quite diverse, including all extended versions of the consent rules.[7] `[See p. 2 in ``Supplementary Note for Reviewers'' at the end of this manuscript, for the definition of the extended consent rules.]`

[5] Office for National Statistics, *Ethnic Group Statistics: A Guide for the Collection and Classification of Ethnicity Data* (2003, p. 9).

[6] The role of this partitioning constraint is discussed more explicitly in our companion paper, Cho and Ju (2014).

[7] We refer readers to Cho and Ju (2014) for more details on the extended model without the partitioning property and the extended consent rules. `[These related materials from Cho and Ju (2014) are copied in ``Supplementary Note for Reviewers'' at the end of this manuscript.]` See also Footnote 11.

In the literature on preference aggregation, Arrow's independence axiom, together with a few fairly mild axioms, implies quite an unequal distribution of decision power: only a single person or a group of persons is decisive (Arrow, 1951; Blau, 1957, 1972; Guha, 1972; Mas-Colell and Sonnenschein, 1972). In the literature on aggregation of equivalence relations, Fishburn and Rubinstein (1986a, 1986b) and Dimitrov, Marchant, and Mishra (2012) consider a variant of Arrow's independence axiom (Fishburn and Rubinstein call it "binary independence") and establish similar results. In contrast to these, our independence axiom admits more diverse power distributions, including both the equal distribution of power as in the liberal rule and the most unequal distribution of power as in the dictatorial rules.

Of particular relevance to our investigation is Miller (2008). He studies binary identification problems in a model where the group whose membership is to be decided can vary. He characterizes the family of one-vote rules (similarly defined in the binary setup) but his results are based on the axiom of "consistency", requiring that decisions across groups be consistent with respect to the conjunction and disjunction of groups ("J and K", "J or K").[8] A proper comparison of our paper and Miller (2008) requires an extended model that subsumes both models. In our companion paper, Cho and Ju (2014), we introduce an extended setup where social decision rules need to identify not only two or more groups but all derived groups that are obtained by conjuntion or disjunction of the basic groups. `[See ''Supplementary Note for Reviewers'' at the end of this manuscript, which provides these related materials in Cho and Ju (2014).]` Using this extended setup, we find that an independence axiom[9], much stronger than our *independence of irrelevant opinions*, is implicitly assumed in Miller (2008) and, together with his consistency axiom, plays a critical role in proof. Although the family of rules our set of axioms characterizes is similar to Miller's (2008), neither his strong independence nor consistency is used in our results. Moreover, *independence of irrelevant opinions*, *non-degeneracy*, and a certain unanimity axiom characterize a larger family

[8] He calls the axiom "separability". Meet separability requires the equivalence of (i) the conjunction of the two decisions for group "J" and group "K"; and (ii) the decision for group "J and K". Join separability requires the equivalence of (i) the disjunction of the two decisions for group "J" and group "K"; and (ii) the decision for group "J or K".

[9] It requires, for instance, that the decision on group "J and K" be independent of the opinions on group J or the opinions on group K, which are quite relevant.

of rules than the one-vote rules Miller (2008) characterizes. [`See Proposition 1 in ``Supplementary Note for Reviewers''.`]

The rest of the paper proceeds as follows. In Section 2, we introduce the model and axioms for group identification. We establish some preliminary results in Section 3. The main characterization results are in Section 4. We conclude with a few remarks in Section 5.

2 The Model

There are n persons, each of whom needs to be identified as a member of one among m groups. Let $\boldsymbol{N} \equiv \{1, \cdots, n\}$ be the set of persons and $\boldsymbol{G} \equiv \{1, \cdots, m\}$ the set of groups. We assume, unless noted otherwise, that $n \geq 2$ and $m \geq 3$. Each person $i \in N$ has an ***opinion*** on who she believes are the members of each of these groups. The opinion is represented by $\boldsymbol{P_i} \equiv (P_{ij})_{j \in N} \in G^N$, where for all $j \in N$ and all $k \in G$, $P_{ij} = k$ when person i views person j as a member of group k. Individual opinions $P_1, \cdots, P_n$ constitute a (identification) ***problem*** $\boldsymbol{P} \equiv (P_{ij})_{i,j \in N}$, an $n \times n$ matrix. Let $\boldsymbol{\mathcal{P}} \equiv G^{N \times N}$ be the set of all problems. A ***domain*** $\boldsymbol{\mathcal{D}} \subseteq \mathcal{P}$ is a non-empty subset of $\mathcal{P}$. We call $\mathcal{P}$ the ***universal domain***. When $m = 2$, our model reduces to the standard, binary group identification model (Kasher and Rubinstein, 1997; Samet and Schmeidler, 2003): in essence, there is only one group and each person has an opinion about who belong to the group.

A ***decision*** is a profile $x \equiv (x_i)_{i \in N} \in G^N$, where, for all $i \in N$ and all $k \in G$, $x_i = k$ means that person i belongs to group k. Given a domain $\mathcal{D}$, a social decision rule, briefly a ***rule*** $f \colon \mathcal{D} \to G^N$ associates with each problem in $\mathcal{D}$ a decision. For example, a ***plurality rule*** puts each person i in the group for which she wins the most votes, namely, group k such that for all $k' \in G$, $|\{j \in N : P_{ji} = k\}| \geq |\{j \in N : P_{ji} = k'\}|$ and when the inequality holds with equality, the tie-breaking condition of $k \leq k'$ is satisfied.[10] Different tie-breaking methods lead to different plurality rules.

In the binary model, the consent rules (Samet and Schmeidler, 2003) allow each person i to determine her own membership if her opinion about herself wins a sufficient

[10]Any linear ordering of the groups can be used as a tie-breaking method.

consent from others (that is, the number of persons agreeing with i, $P_{ji} = P_{ii}$, is no less than a certain quota). When the consent quota equals the minimum level of 1, everyone self-determines her own membership; i.e., for all $P \in \mathcal{D}$ and all $i \in N$, i belongs to group P_{ii}. This rule is called the ***liberal rule***. When a consent rule is not liberal (the consent quota is at least 2), it is possible that a person does not win a sufficient consent from others for the group she claims to be a member of. With the insufficient consent, she fails to self-determine her membership, which in the binary model, means that she belongs to the other group. In our multinary model, this case of insufficient consent is indeterminate since there are two or more other groups.[11] Hence, none of the consent rules except for the liberal rule is well-defined in our model.

Nevertheless, we can define similar rules using a mapping $\delta \colon G \to G$, associating with all $k \in G$ the ***default decision against group k***, denoted by $\boldsymbol{\delta_k} \in G$. Group δ_k serves as the default membership for a person who considers herself belonging to group k but fails to win a sufficient consent from others. For all $k \in G$, let $\boldsymbol{q_k} \in \mathbb{N}$ be the **consent quota for group k**. The ***consent rule with default decisions $\boldsymbol{\delta \equiv (\delta_k)_{k \in G}}$ and quotas $\boldsymbol{q \equiv (q_k)_{k \in G}}$***, denoted by $\boldsymbol{f^{\delta,q}}$, is defined as follows: for all $P \in \mathcal{D}$ and all $i \in N$ with $P_{ii} = k$,

(i) if $|\{j \in N : P_{ji} = k\}| \geq q_k$, then $f_i^{\delta,q}(P) = k$; and

(ii) otherwise, $f_i^{\delta,q}(P) = \delta_k$.[12]

Under the consent rule $f^{\delta,q}$, each person $i \in N$ belongs to either the group of her own decision (P_{ii}) or the opposite ($\delta_{P_{ii}}$). Clearly, when $q_1 = \cdots = q_m = 1$, $f^{\delta,q}$ coincides with the liberal rule, whatever δ is.

[11] In order to allow for this indeterminacy, we may expand G to include "non-k" ($\neg k$) for each $k \in G$. Let $G_{ex} \equiv G \cup \{\neg k : k \in G\}$. A *quasi-decision* is a profile $x \in G_{ex}^N$, possibly admitting an indeterminate decision for some person, such as "non-k", and a *quasi-rule* is a mapping $f \colon \mathcal{P} \to G_{ex}^N$. The consent rules by Samet and Schmeidler (2003) are examples of quasi-rules. Among them, only the liberal rule is well-defined in our multinary model. It is evident that all consent quasi-rules satisfy *independence of irrelevant opinions* (to be defined later) as well as *non-degeneracy*. Thus, our main results do not hold for these quasi-rules.

[12] Our definition permits the possibility that for some $k \in G$, $\delta_k = k$. For such k, whenever $P_{ii} = k$, $f^{\delta,q}$ puts person i in group k. Now given (δ, q), define (δ', q') as follows: for all $k \in G$ such that $\delta_k = k$, $\delta'_k \neq k$ and $q'_k = 1$; for all other $k \in G$, $\delta'_k = \delta_k$ and $q'_k = q_k$. Then $f^{\delta,q}$ and $f^{\delta',q'}$ are equivalent. Also, note that in the binary model by Samet and Schmeidler (2003), our definition coincides with their definition of consent rules once $q_k + q_{\delta_k} \leq n + 2$ is added, which is needed for their monotonicity axiom.

When there is a status quo group $\kappa \in G$ where all persons initially belong, one can define a consent rule that determines regrouping of all members in the status quo group by setting, for all $k \in G$, $\delta_k = \kappa$. Then each person either belongs to the group of her own decision (P_{ii}) or the status quo group (κ). Thus when she considers herself to be in the status quo group, her opinion is decisive for her own membership. She needs others' consent only when she considers herself not belonging to the status quo group. We discuss an extension of this idea and other related issues in Section 5

Axioms

Should a person belong to J because many others believe that she belongs to K rather than to L? Should the membership for a certain group depend on opinions on the other groups? The answer, obviously, will differ from context to context. Nevertheless, when there is no concensus on how the groups are interrelated or no objective basis for judging the correlation among the groups, any decision rule relying on a particular correlation may incur social controversy, hard to be resolved with an agreement and possibly leading to conflicts among groups. Our main axiom requires cross-group independence. Consider two problems, P and P', such that all persons agree on the membership for group k; that is, each person i considers each person j to be a member of group k at P if and only if she does so at P'. They may differ on the membership for other groups but if this difference is viewed as irrelevant when making a decision on who belong to group k, it is natural to require that $f(P)$ and $f(P')$ agree on the group-k membership.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{D}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$, $f_i(P) = k$ if and only if $f_i(P') = k$.

It is evident that the liberal rule satisfies *independence of irrelevant opinions.* There are other rules as simple as the liberal rule, which also satisfy this axiom. They are defined in Section 4. The consent rules with default decisions do not necessarily satisfy the independence axiom; the membership for the default group against group k relies on the opinions on group k. Nevertheless, on some restricted domains, they may do so. Here is an example.

Example 1. Suppose that group $\nu \in G$ is a null group (non-membership for any of the other groups in $G \backslash \{\nu\}$) and consider the domain $\mathcal{D}_\nu \equiv \{P \in \mathcal{P}$: for all $i, j \in N$, $P_{ij} \in G \backslash \{\nu\}\}$ where all persons classify everyone into non-null groups in $G \backslash \{\nu\}$ (they all used to be just "Earthians" but now need to be classified into several subgroups in $G \backslash \{\nu\}$, one of which they believe everyone belongs to; so ν is the group of Earthians who are decided not to be a member of any of the subgroups in $G \backslash \{\nu\}$). On this domain $\mathcal{D}_\nu$, all consent rules with the constant default decision of ν satisfy *independence of irrelevant opinions.* This is because all problems in $\mathcal{D}_\nu$ share the common opinion on the null group ν: no one ever believes anyone, including herself, to be in the null group. Thus, with regard to this null group, *independence of irrelevant opinions* has no bite; with regard to the other non-null groups $k \in G \backslash \{\nu\}$, it is evident from the definition that the membership for group k depends only on the opinions on group k (who they believe belong to k or not). $\triangle$

We also consider the following fairly standard axioms in the group identification literature. Given a permutation $\pi \colon N \to N$, for all $P \in \mathcal{P}$, let $\boldsymbol{P_\pi} \equiv (P_{\pi(i),\pi(j)})_{i,j \in N}$ and $\boldsymbol{f_\pi(P)} \equiv (f_{\pi(i)}(P))_{i \in N}$. Permutation π changes the names of persons and P_π is the problem obtained by changing names through π. Name changes are only nominal and shifts no fundamental content. Thus, it is reasonable to require that the decision be unaffected by such nominal changes (Samet and Schmeidler, 2003).

Symmetry. For all $P \in \mathcal{D}$ and all permutations $\pi \colon N \to N$ such that $P_\pi \in \mathcal{D}$, $f(P_\pi) = f_\pi(P)$.

Our next axiom concerns the decision for "unanimous" opinion profiles: if all persons consider all persons belonging to one group, say group k, then all persons should be classified into group k.

Unanimity. For all $k \in G$ such that $k_{n \times n} \in \mathcal{D}$, $f(k_{n \times n}) = k_{1 \times n}$.

A rule may be "degenerate" for a person in that there is one fixed group into which the rule always classifies her, regardless of opinions. We require that such degeneracy occur for no person. Clearly, this is weaker than *unanimity.*

Non-Degeneracy. For all $i \in N$, there are $P, P' \in \mathcal{D}$ such that $f_i(P) \neq f_i(P')$.

3 Independence of Irrelevant Opinions and Decomposability

A problem contains binary information on the memberships for all groups. Thus, we may "decompose" the problem into multiple binary problems, obtain binary decisions for the latter, and combine them into a single decision. The combined decision may or may not be the same as the decision a rule assigns to the initial problem. In this section, we establish that *independence of irrelevant opinions* is almost equivalent to requiring that the two decisions be the same.

More precisely, let $\mathcal{B} \equiv \{0,1\}^{N \times N}$. Given $P \in \mathcal{P}$, for all $k \in G$, let $\boldsymbol{B^{P,k}} \in \mathcal{B}$ be the binary problem concerning group k derived from P; i.e., for all $i, j \in N$, (i) if $P_{ij} = k$, then $B^{P,k}_{ij} = 1$; and (ii) if $P_{ij} \neq k$, then $B^{P,k}_{ij} = 0$. An (binary) ***approval function*** $\varphi\colon \mathcal{B} \to \{0,1\}^N$ associates with each binary problem $B \in \mathcal{B}$ a binary decision, namely, a profile of 0's and 1's, where for all $i \in N$, $\varphi_i(B) = 0$ means the disapproval of i's membership and $\varphi_i(B) = 1$ means the approval of i's membership. For all binary problems $B \in \mathcal{B}$, let $\bar{\boldsymbol{B}} \equiv 1_{n\times n} - B$ be the dual problem of B. Likewise, for all binary decisions $x \in \{0,1\}^N$, let $\bar{\boldsymbol{x}} \equiv 1_{1\times n} - x$ be the dual decision of x.

Using these definitions, each problem $P \in \mathcal{P}$ can be decomposed into m binary problems, $B^{P,1}, \cdots, B^{P,m}$. The next axiom requires that the decision for problem P be identical to the combination of m binary decisions for the m binary problems assigned by an approval function.

Decomposability. There is an approval function φ such that for all $P \in \mathcal{D}$, all $i \in N$, and all $k \in G$, $f_i(P) = k$ if and only if $\varphi_i(B^{P,k}) = 1$.

In this case, we say that ***$\boldsymbol{f}$ is represented by $\boldsymbol{\varphi}$***.

We show that an approval function representing a *decomposable* rule satisfies the following properties. Approval function φ is ***m-unit-additive*** if for all m binary problems $B^1, \cdots, B^m \in \mathcal{B}$,

$$\sum_{k\in G} B^k = 1_{n\times n} \text{ implies } \sum_{k \in G} \varphi(B^k) = 1_{1\times n}. \tag{1}$$

It is ***unanimous*** if $\varphi(0_{n\times n}) = 0_{1\times n}$ and $\varphi(1_{n\times n}) = 1_{1\times n}$. The ***dual of $\boldsymbol{\varphi}$***, denoted

$\boldsymbol{\varphi^d}$, is the approval function such that for all $B \in \mathcal{B}$, $\varphi^d(B) = \overline{\varphi(\overline{B})}$. We say that φ is ***self-dual*** if $\varphi = \varphi^d$. Finally, φ is ***monotonic*** if for all $B, B' \in \mathcal{B}$ such that $B \leq B'$, $\varphi(B) \leq \varphi(B')$.

Proposition 1. *Consider the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). An approval function represents a* decomposable *rule if and only if it is m-*unit-additive*. Also, if an approval function is m-*unit-additive*, then it is* unanimous, self-dual, *and* monotonic.

Proof. First, we prove the "if and only if" statement. Note that for all $P \in \mathcal{P}$, $\sum_{k\in G} B^{P,k} = 1_{n\times n}$ and that for all $B^1, \cdots, B^m \in \mathcal{B}$ with $\sum_{k\in G} B^k = 1_{n\times n}$, there is $P \in \mathcal{P}$ such that $B^1 = B^{P,1}, \cdots, B^m = B^{P,m}$. This observation is enough to prove the "if" part. Next, suppose that an approval function φ can represent a *decomposable* rule. Then for all $P \in \mathcal{P}$ and all $i \in N$, there is exactly one $k \in G$ with $\varphi_i(B^{P,k}) = 1$. Since for all $k' \in G\backslash\{k\}$, $\varphi_i(B^{P,k'}) = 0$, $\sum_{k\in G} \varphi_i(B^{P,k}) = 1$. Thus, φ is *m-unit-additive.*

Assume, henceforth, that φ is *m-unit-additive.* To prove that φ is *unanimous*, let $i \in N$. Let $B^1 \equiv 1_{n\times n}$, and for all $k \in G\backslash\{1\}$, $B^k \equiv 0_{n\times n}$. Let $s \equiv \varphi_i(1_{n\times n})$ and $t \equiv \varphi_i(0_{n\times n})$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, $1 = \sum_{k\in G} \varphi_i\left(B^k\right) = s + (m-1)t$. Since $s, t \in \{0, 1\}$ and $m \geq 3$, it follows that $s = 1$ and $t = 0$.

To prove that φ is *self-dual*, let $B \in \mathcal{B}$. Let $B^1 \equiv B$, $B^2 \equiv \overline{B}$, and for all $k \in G\backslash\{1, 2\}$, $B^k \equiv 0_{n\times n}$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, then by *$m$-unit-additivity* and *unanimity*, $1_{1\times n} = \sum_{k\in G} \varphi\left(B^k\right) = \varphi(B) + \varphi(\overline{B})$. This gives $\varphi(B) = \overline{\varphi(\overline{B})}$.

Finally, to prove that φ is *monotonic*, let $B, B' \in \mathcal{B}$ be such that $B \leq B'$. Let $i \in N$. If $\varphi_i(B) = 0$, then trivially, $\varphi_i(B) \leq \varphi_i(B')$. Thus, assume that $\varphi_i(B) = 1$. Let $B^1 = B$ and $B^2 = \overline{B'}$. Let $B^3, \cdots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$ (such $B^3, \cdots, B^m$ exist because $\overline{B} \geq \overline{B'}$ and $B^1 + B^2 = B + \overline{B'} \leq 1_{n\times n}$). Since $\sum_{k\in G} \varphi_i\left(B^k\right) = 1$ and $\varphi_i\left(B^1\right) = 1$, $0 = \varphi_i\left(B^2\right) = \varphi_i(\overline{B'})$. Since φ is *self-dual*, $\varphi_i(B') = \overline{\varphi_i(\overline{B'})} = 1$. □

If f is *independent of irrelevant opinions*, then it can be "represented" by m approval functions $(\varphi^k)_{k\in G}$. To see this, let $k \in G$. Define an approval function $\varphi^k : \mathcal{B} \to \{0, 1\}^N$ as follows: for all $B \in \mathcal{B}$ and all $i \in N$, $\varphi^k_i(B) = 1$ if and only if for some $P \in \mathcal{P}$, $B^{P,k} = B$ and $f_i(P) = k$. Clearly, by *independence of irrelevant opinions*, φ^k is well-defined[13], and f is represented by $(\varphi^k)_{k\in G}$; i.e., for all $P \in \mathcal{P}$, all $i \in N$, and all $k \in G$,

[13] By *independence of irrelevant opinions*, for all $P, P' \in \mathcal{P}$ such that $B^{P,k} = B^{P',k} = B$, $f_i(P) = k$ if and only if $f_i(P') = k$.

$f_i(P) = k$ if and only if $\varphi_i^k(B^{P,k}) = 1$. When all the approval functions are identical ($\varphi^1 = \cdots = \varphi^m$), f is *decomposable*. Therefore, *decomposability* implies *independence of irrelevant opinions*. The converse does not hold. As we show below, the essential difference between the two axioms is *non-degeneracy*. To prove it, we use the following lemma.

Lemma 1. *On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$),* independence of irrelevant opinions *and* non-degeneracy *together imply* unanimity.

Proof. Let f be a rule satisfying *independence of irrelevant opinions* and *non-degeneracy*. Then there are approval functions $(\varphi^k)_{k\in G}$ representing f. Now we proceed in three steps.

Step 1: *For all $i \in N$, all $P \in \mathcal{P}$, and all $\ell \in G\backslash\{f_i(P)\}$, $\varphi_i^\ell(0_{n\times n}) = 0$.*

Let $i \in N$ and $P \in P$. Let $k \equiv f_i(P)$. Let $\ell, h \in G\backslash\{k\}$ be distinct. Let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P'_{jj'} = k$ if and only if $P_{jj'} = k$; and (ii) $P'_{jj'} = h$ if and only if $P_{jj'} \neq k$. By *independence of irrelevant opinions*, $f_i(P') = f_i(P) = k$, so that $f_i(P') \neq \ell$. Then $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{P',\ell}) = 0$.

Step 2: *For all $i \in N$ and all $k \in G$, $\varphi_i^k(0_{n\times n}) = 0$.*

Let $i \in N$. By *non-degeneracy*, there are $P, P' \in \mathcal{P}$ such that $f_i(P) \neq f_i(P')$. Let $k \equiv f_i(P)$ and $\ell \equiv f_i(P')$. By Step 1, for all $h \in G\backslash\{k\}$, $\varphi_i^h(0_{n\times n}) = 0$. Similarly, for all $h \in G\backslash\{\ell\}$, $\varphi_i^h(0_{n\times n}) = 0$.

Step 3: *f is* unanimous*.*

Suppose, by contradiction, that for some $i \in N$ and $k \in G$, $f_i(k_{n\times n}) \neq k$. Let $\ell \equiv f_i(k_{n\times n})$. Then $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{k_{n\times n},\ell}) = 1$, contradicting Step 2. □

Now we prove the logical relation between *independence of irrelevant opinions* and *decomposability*.

Proposition 2. *On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$), the combination of* independence of irrelevant opinions *and* non-degeneracy *is equivalent to* decomposability*.*

Proof. We already noted that *decomposability* implies *independence of irrelevant opinions*. When a rule is *decomposable*, by Proposition 1, the approval function representing it is *unanimous*. Therefore, the rule is also *unanimous*, and so *non-degenerate*.

To prove the converse, let f be a rule satisfying *independence of irrelevant opinions* and *non-degeneracy*. Then f is represented by a profile of m approval functions $(\varphi^k)_{k \in G}$. By Lemma 1, f is *unanimous*. Now we proceed in two steps.

Step 1: *For all $i \in N$ and all $P \in \mathcal{P}$, $f_i(P)$ is one of the entries of P.*

Suppose, by contradiction, that for some $i \in N$ and $P \in \mathcal{P}$, $f_i(P)$ is not one of the entries of P; i.e., for some k such that $B^{P,k} = 0_{n\times n}$, $f_i(P) = k$. Let $\ell \in G$ be one of the entries of P and consider $\ell_{n\times n} \in \mathcal{P}$. Then $B^{P,k} = 0_{n\times n} = B^{\ell_{n\times n},k}$. Thus applying *independence of irrelevant opinions* to P and $\ell_{n\times n}$, $f_i(P) = k$ implies $f_i(\ell_{n\times n}) = k$, which contradicts *unanimity*.

Step 2: $\varphi^1 = \varphi^2 = \cdots = \varphi^m$.

Suppose, by contradiction, that there are $k, \ell \in G$ such that $\varphi^k \neq \varphi^\ell$. Then there are $B \in \mathcal{B}$ and $i \in N$ such that $\varphi_i^k(B) \neq \varphi_i^\ell(B)$. Without loss of generality, assume that $\varphi_i^k(B) = 0$ and $\varphi_i^\ell(B) = 1$. Let $h \in G \backslash \{k, \ell\}$. Let $P \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P_{jj'} = k$ if and only if $B_{jj'} = 1$. Similarly, let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P'_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P'_{jj'} = \ell$ if and only if $B_{jj'} = 1$. By construction, $B^{P,k} = B^{P',\ell} = B$. Since $\varphi_i^k(B) = 0$ and $\varphi_i^\ell(B) = 1$, it follows that $f_i(P) \neq k$ and $f_i(P') = \ell$. By Step 1, $f_i(P) \neq k$ implies $f_i(P) = h$. Note that $B^{P,h} = B^{P',h}$. Hence, applying *independence of irrelevant opinions* to P and P', $f_i(P) = h$ implies $f_i(P') = h$, which contradicts $f_i(P') = \ell$. □

Note that by Proposition 2 and Lemma 1, *decomposability* also implies *unanimity*.

4 Main Results

In this section, we present our main characterization results. We first characterize the rules satisfying *independence of irrelevant opinions* and *non-degeneracy*. These rules are represented by the "one-vote" approval functions that Miller (2008) introduces in the binary identification model. An approval function φ is a ***one-vote approval function*** if for all $i \in N$, there are $j, h \in N$ such that for all $B \in \mathcal{B}$, $\varphi_i(B) = B_{jh}$. A rule f is a ***one-vote rule*** if for all $i \in N$, there are $j, h \in N$ such that for all $P \in \mathcal{P}$,

$f_i(P) = P_{jh}$. The one-vote rules are *decomposable*, represented by one-vote approval functions; moreover, they are the only *decomposable* rules.

Theorem 1. *On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$), the following are equivalent.*
(i) *A rule is* independent of irrelevant opinions *and* non-degenerate;
(ii) *A rule is* decomposable;
(iii) *A rule is a one-vote rule.*

Proof. By Proposition 2, we only have to show the equivalence of (ii) and (iii). We only prove the non-trivial implication, "(ii) implies (iii)". Consider a *decomposable* rule represented by an approval function φ. It suffices to show that φ is a one-vote approval function. We use the following notation in the proof. Let $B \in \mathcal{B}$. Let $|\boldsymbol{B}| \equiv \sum_{i,j \in N} B_{ij}$ be the number of 1's in B. Also, B is a ***unit binary problem*** if $|B| = 1$. For all $i, j \in N$, let $U^{ij} \in \mathcal{B}$ be the unit binary problem such that $U^{ij}_{ij} = 1$. Now let $i \in N$. We proceed in two steps.

Step 1: *There are $j, h \in N$ such that $\varphi_i\left(U^{jh}\right) = 1$.*

By Proposition 1, φ is *m-unit-additive*, *unanimous*, *self-dual*, and *monotonic*. Suppose, by contradiction, that for all $B \in \mathcal{B}$,

$$|B| = 1 \text{ implies } \varphi_i(B) = 0. \tag{2}$$

We prove by induction that for all $B \in \mathcal{B}$, $\varphi_i(B) = 0$.

Let $\ell \in \mathbb{N}$ be such that $\ell < n^2$ and assume that for all $B \in \mathcal{B}$,

$$|B| \leq \ell \text{ implies } \varphi_i(B) = 0. \tag{3}$$

Let $B \in \mathcal{B}$ be such that $|B| = \ell + 1$. Then $|\bar{B}| = n^2 - \ell - 1$ and there are B^1 and B^2 such that $|B^1| = 1$, $|B^2| = \ell$ and $B^1 + B^2 + \bar{B} = 1_{n \times n}$. By *m-unit-additivity* and *unanimity*, $\varphi_i(B^1) + \varphi_i(B^2) + \varphi_i(\bar{B}) = 1$. Since by the induction hypothesis (3), $\varphi_i(B^1) = \varphi_i(B^2) = 0$, we obtain $\varphi_i(\bar{B}) = 1$. By *self-duality*, $\varphi_i(B) = 0$. Hence, for all $B \in \mathcal{B}$,

$$|B| \leq \ell + 1 \text{ implies } \varphi_i(B) = 0.$$

Therefore, (2) and the induction argument prove that for all $B \in \mathcal{B}$, $\varphi_i(B) = 0$. In particular, $\varphi_i(1_{n\times n}) = 0$, which contradicts *unanimity* of φ.

Step 2: *For all $B \in \mathcal{B}$, $\varphi_i(B) = 1$ if and only if $B_{jh} = 1$.*

Let $j, h \in N$ be such that $\varphi_i(U^{jh}) = 1$. Let $B \in \mathcal{B}$. If $B_{jh} = 1$, then since $B \geq U^{jh}$, *monotonicity* implies that $\varphi_i(B) \geq \varphi_i\left(U^{jh}\right) = 1$. If $B_{jh} = 0$, then since $B \leq \overline{U^{jh}}$, *monotonicity* and *self-duality* imply that $\varphi_i(B) \leq \varphi_i\left(\overline{U^{jh}}\right) = 0$. □

When $n \geq 3$, among the one-vote rules, there is only one *symmetric* rule: the liberal rule.

Theorem 2. *Assume that there are at least three persons ($n \geq 3$). On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$), the following are equivalent.*
(i) *A rule is* independent of irrelevant opinions*,* non-degenerate*, and* symmetric;
(ii) *A rule is* decomposable *and* symmetric;
(iii) *A rule is the liberal rule.*

Proof. We prove that (ii) implies (iii). Let f be a rule satisfying *decomposability* and *symmetry*. By Theorem 1, it is a one-vote rule. Then there is a function $h\colon N \to N \times N$ such that for all $P \in \mathcal{P}$ and all $i \in N$, $f_i(P) = P_{h(i)}$. Now *symmetry* implies that h satisfies the following: for all permutations $\pi : N \to N$ and all $i \in N$,

$$h(\pi(i)) = (\pi(h_1(i)), \pi(h_2(i))) . \tag{4}$$

It is enough to show that for all $i \in N$, $h(i) = (i, i)$. Now suppose, by contradiction, that there is $i \in N$ such that $h(i) \neq (i, i)$. Let $(j, k) \equiv h(i)$. Without loss of generality, assume that $k \neq i$. Since $n \geq 3$, there is $\ell \in N \backslash \{i, k\}$. Let $\pi\colon N \to N$ be the transposition of k and ℓ. Then $h(\pi(i)) = h(i) = (j, k)$ but $(\pi(h_1(i)), \pi(h_2(i))) = (\pi(j), \pi(k)) = (j, \ell)$, contradicting (4). □

Remark 1. When $n = 2$, there are other, non-liberal one-vote rules satisfying the axioms in Theorem 2. In fact, parts (i) and (ii) of Theorem 2 are equivalent to the following statement: (iii′) the rule f is (a) the liberal rule; or (b) such that for all $P \in \mathcal{P}$, $f(P) = (P_{21}, P_{12})$; or (c) such that for all $P \in \mathcal{P}$, $f(P) = (P_{12}, P_{21})$; or (d) such that for all $P \in \mathcal{P}$, $f(P) = (P_{22}, P_{11})$. Therefore, when there are only two persons, four rules satisfy the axioms in parts (i) or (ii). △

Proof of Remark 1. Let $n = 2$. As in the proof of Theorem 2, we can obtain a function $h : N \to N \times N$. By *symmetry*, h satisfies equation (4). Since $n = 2$, $h(1)$ determines $h(2)$ as well: letting $\pi : N \to N$ be the transposition of 1 and 2, it follows that $h(2) = h(\pi(1)) = (\pi(h_1(1)), \pi(h_2(1)))$. For instance, if $h(1) = (1, 2)$, then $h(2) = (2, 1)$. Since $h(1) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, we can define h in four different ways, thus obtaining the four rules in Remark 1. □

Remark 2. On the restricted domain in Example 1, there are numerous consent rules, far from being liberal, which satisfy all axioms in parts (i) and (ii). The logical independence of the three axioms in part (i) (also the two axioms in part (ii)) are established by the following examples: one-vote rules (satisfying all but *symmetry*), constant "uniform identification" rules (satisfying all but *non-degeneracy*),[14] and plurality rules (satisfying all but *independence of irrelevant opinions*). △

5 Concluding Remarks

In the binary group identification model, *independence of irrelevant opinions* is vacuous; *decomposability* is also mild since it coincides with *self-duality*. However, with three or more groups, the two axioms become very demanding as shown by Theorems 1 and 2. The contrasting consequences of these axioms in the binary and multinary setups are similar to those in Arrovian preference aggregation with two alternatives and that with three or more alternatives.[15]

We can extend our model to allow for "status-quo memberships", and modify independence accordingly. Consider a mapping $\sigma\colon N \to G$, associating with each person $i \in N$ her ***status quo membership*** $\boldsymbol{\sigma_i} \in G$. For all $k \in G$, let $\boldsymbol{q_k} \in \mathbb{N}$ be the **consent quota for group *k***. The ***regrouping consent rule with the status quo*** $\boldsymbol{\sigma \equiv (\sigma_i)_{i \in N}}$ ***and quotas*** $\boldsymbol{q \equiv (q_k)_{k \in G}}$, denoted by $\boldsymbol{f^{\sigma,q}}$, is defined as follows: for all $P \in \mathcal{P}$ and all $i \in N$ with $P_{ii} = k$,

[14] A group, say group k is fixed and everyone always belongs to group k.

[15] When there are three or more alternatives, independence of irrelevant alternatives, transitivity, and unanimity (or Pareto principle) imply dictatorship (Arrow's Impossibility Theorem; Arrow, 1951). When there are two alternatives, independence of irrelevant alternatives and transitivity are vacuous, and there are numerous non-dictatorial aggregation rules that perform well in terms of, e.g., "monotonicity" and "anonymity".

(i) if $|\{j \in N : P_{ji} = k\}| \geq q_k$, then $f_i^{\sigma,q}(P) = k$; and

(ii) otherwise, $f_i^{\sigma,q}(P) = \sigma_i$.

Thus each person $i \in N$ only belongs to the group of her own decision (P_{ii}) or her status quo group (σ_i). She can always decide to stay at the status quo; she needs others' consent only when she claims for a change. Although these rules are similar to the consent rules with default decisions, the two families are different. For instance, in the binary model, the regrouping consent rules do not coincide with the consent rules by Samet and Schmeidler (2003).

To take account of the status quo memberships, *independence of irrelevant opinions* can be relaxed by requiring the same independence only for those groups to which persons do not initially belong. Thus, only when a person is assigned to a new group, the membership decision for that group should be independent of opinions on the other groups.

Regrouping Independence. Let $P, P' \in \mathcal{P}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$ with $\sigma_i \neq k$, $f_i(P) = k$ if and only if $f_i(P') = k$.

Regrouping independence weakens *independence of irrelevant opinions* only slightly. But interestingly, it is satisfied by the regrouping consent rules, which is a substantially larger family of rules than the one-vote rules. Also, the regrouping consent rules satisfy *unanimity* (hence, *non-degeneracy*) and if for all $i, j \in N$, $\sigma_i = \sigma_j$, they also satisfy *symmetry*. When $q_1 = \cdots = q_m = 1$, $f^{\sigma,q}$ coincides with the liberal rule.

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Supplementary Note for Reviewers

Extended Model and Applications

We now consider the extended framework where the issues arise not only of identifying who is a J and who is a K, but also of simultaneously identifying who is a "J and K", who is a "J or K", who is a "non-J and K", who is a "J and non-K", and who is a "non-J and non-K". A reduced form of this model focusing on binary identification is studied by Miller (2008). Here we extend his model to deal with multinary identification.

In what follows, we call G the set of ***basic groups***. Basic groups are not necessarily mutually exclusive (i.e., a person can belong to two or more of them). Let $\mathcal{G}$ be the Boolean algebra generated by the set of basic groups G through conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$). Let $\mathfrak{1} \in \mathcal{G}$ be the group "everyone" and $\mathfrak{0} \in \mathcal{G}$ the group "no one". We call the groups in $\mathcal{G} \backslash G$ ***derived groups***. A derived group is ***complete*** if it is a conjunction of qualification or disqualification for all basic groups in G. For example, when $G = \{a, b\}$, there are four complete groups, which are $a \wedge b$, $\neg a \wedge b$, $a \wedge \neg b$, and $\neg a \wedge \neg b$. Let $\mathcal{G}^c$ be the set of complete groups. Complete groups are mutually exclusive and they cover the whole set N of persons.

An ***extended problem*** $\mathbf{B} \equiv (\mathbf{B}^g)_{g\in\mathcal{G}} \in \mathcal{B}^{\mathcal{G}}$ is a profile of binary problems for all groups in $\mathcal{G}$ satisfying the following property: for all $g, g' \in \mathcal{G}$, $\mathbf{B}^{g\wedge g'} = \mathbf{B}^g \wedge \mathbf{B}^{g'}$, $\mathbf{B}^{g\vee g'} = \mathbf{B}^g \vee \mathbf{B}^{g'}$, $\mathbf{B}^{\mathfrak{1}} = 1_{n\times n}$, and $\mathbf{B}^{\mathfrak{0}} = 0_{n\times n}$ (so $\mathbf{B}^{\neg g} = 1_{n\times n} - \mathbf{B}^g$).[1] Call this property ***opinion-consistency***. Let $\mathfrak{B}$ be the set of all extended problems. Note that by *opinion-consistency*, a profile $(\mathbf{B}^g)_{g\in G}$ of binary problems for all basic groups ***generates*** an (unique) extended problem. An ***extended decision*** $x \equiv (x^g)_{g\in\mathcal{G}} \in (\{0,1\}^N)^{\mathcal{G}}$ is a profile of decisions for all groups in $\mathcal{G}$ with $x^{\mathfrak{1}} = 1_{1\times n}$ and $x^{\mathfrak{0}} = 0_{1\times n}$. An extended decision $x \equiv (x^g)_{g\in\mathcal{G}}$ satisfies ***meet-consistency*** if for all $g, g' \in \mathcal{G}$, $x^{g\wedge g'} = x^g \wedge x^{g'}$. It satisfies ***join-consistency*** if for all $g, g' \in \mathcal{G}$, $x^{g\vee g'} = x^g \vee x^{g'}$. Let $(x^k)_{k\in G}$ be a profile of decisions for all basic groups and call it a ***basic group decision***. Then $(x^k)_{k\in G}$ ***generates*** an (unique) extended decision satisfying the two *consistency* properties.

[1]Notation: $\mathbf{B}^g \wedge \mathbf{B}^{g'} \equiv \left(\min\{\mathbf{B}^g_{ij}, \mathbf{B}^{g'}_{ij}\}\right)_{i,j\in N}$ and $\mathbf{B}^g \vee \mathbf{B}^{g'} \equiv \left(\max\{\mathbf{B}^g_{ij}, \mathbf{B}^{g'}_{ij}\}\right)_{i,j\in N}$. For all $x^g, x^{g'} \in \{0,1\}^N$, $x^g \wedge x^{g'}$ and $x^g \vee x^{g'}$ are similarly defined.

An ***extended rule*** $F : \mathfrak{B} \rightarrow \left(\{0,1\}^N\right)^{\mathcal{G}}$ maps each extended problem into an extended decision. For all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, let $F^g(\mathbf{B})$ be the group-g decision for $\mathbf{B}$ assigned by the extended rule F. The extended rule F satisfies ***meet-consistency*** (or ***join-consistency***, respectively) if it always produces a *meet-consistent* (or *join-consistent*) decision; i.e., for all $\mathbf{B} \in \mathfrak{B}$ and all $g, g' \in \mathcal{G}$, $F^{g \wedge g'}(\mathbf{B}) = F^g(\mathbf{B}) \wedge F^{g'}(\mathbf{B})$ (or $F^{g \vee g'}(\mathbf{B}) = F^g(\mathbf{B}) \vee F^{g'}(\mathbf{B})$).[2] It satisfies ***consistency*** if it satisfies both *meet-consistency* and *join-consistency*. It satisfies ***non-degeneracy*** if for all $g \in \mathcal{G} \backslash \{\mathbf{1}, \mathbf{0}\}$ and all $i \in N$, there are $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^g(\mathbf{B}) = 1$ and $F_i^g(\mathbf{B}') = 0$.

An extended rule F is an ***extended one-vote rule*** if for all $i \in N$, there are $j, h \in N$ such that for all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, $F_i^g(\mathbf{B}) = \mathbf{B}_{jh}^g$. Clearly, all extended one-vote rules are *consistent*. In fact, they are outliers of the very rich family of *consistent* rules. A simple way of constructing a *consistent* extended rule is first to define a decision rule for basic groups—let us call it a ***basic group rule***—and second to extend the basic group decisions to an extended decision using *consistency*. For example, we may use the family of consent approval functions for basic group rules (Samet and Schmeidler, 2003). Let $s,t \in \{1, \cdots, n\}$ with $s + t \leq n + 2$. The ***basic group consent rule with*** $(\boldsymbol{s}, \boldsymbol{t})$ satisfies the following: for all $\mathbf{B} \in \mathfrak{B}$, all $k \in G$, and all $i \in N$, (i) when $\mathbf{B}_{ii}^k = 1$, $F_i^k(\mathbf{B}) = 1$ if and only if $|\{j \in N : \mathbf{B}_{ji}^k = 1\}| \geq s$, and (ii) when $\mathbf{B}_{ii}^k = 0$, $F_i^k(\mathbf{B}) = 0$ if and only if $|\{j \in N : \mathbf{B}_{ji}^k = 0\}| \geq t$. An ***extended consent rule*** is the *consistent* extension of a basic group consent rule. Thus, by construction, all extended consent rules are *consistent*.[3]

The extended consent rules show by example that in the extended setup, *consistency* and *non-degeneracy* do not characterize the family of extended one-vote rules. This may appear to be in conflict with Miller (2008). However, in Miller's (2008) model, another independence axiom, which we define below, is implicit.[4] Once the independence axiom is imposed in addition, the same characterization as in Miller (2008) holds. Yet the independence axiom is stronger than *independence of irrelevant opinions*, and we show that in fact, a weaker set of axioms suffices for the characterization.

[2] Miller (2008) calls these two properties meet separability and join separability.

[3] Alternatively, given (s, t), one may define a rule by applying conditions (i) and (ii) in the definition of the basic group consent rule to all groups $g \in \mathcal{G}$, basic and derived. However, this rule is not *consistent* unless $s = t = 1$, in which case, the rule is just the liberal rule.

[4] We elaborate on this claim after we define *component-wise independence*.

Now we introduce axioms for the extended setup. First is a straightforward extension of *independence of irrelevant opinions*:

Independence of Irrelevant Opinions. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $k \in G$, if $\mathbf{B}^k = \mathbf{B}'^k$, then $F^k(\mathbf{B}) = F^k(\mathbf{B}')$.

Note that this axiom does not require independence across derived groups and so the decision on $a \wedge b$ may depend on B^a and B^b as well as on $B^{a\wedge b}$. It is evident by definition that all extended consent rules are *independent of irrelevant opinions*.

Non-degeneracy in the multinary setup corresponds to the following axiom in the extended setup.

Basic Group Non-Degeneracy. For all $k \in G$ and all $i \in N$, there exist $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^k(\mathbf{B}) = 0$ and $F_i^k(\mathbf{B}') = 1$.

In our model, a decision associates with each person a membership for exactly one basic group. In the extended model, this can be stated as follows.

Basic Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, $\left(\{i \in N : F_i^k(\mathbf{B}) = 1\}\right)_{k\in G}$ is a partition of N.

In the presence of *consistency*, this implies that for all distinct pairs $k, \ell \in G$, $F^{k\wedge\ell}(\cdot)$ is degenerate, taking the constant value of $0_{1\times n}$. This partitioning requirement may be too strong in the extended model and a weaker version may be formulated by requiring similar partitioning only when all persons agree with partitioning by basic groups.

Unanimous Basic Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, if each $i \in N$ partitions N into basic groups at $\mathbf{B}$ (i.e., for all $i \in N$, $(\{j \in N \colon \mathbf{B}_{ij}^k = 1\})_{k\in G}$ is a partition of N), then $\left(\{i \in N : F_i^k(\mathbf{B}) = 1\}\right)_{k\in G}$ is a partition of N.

In our model, each person believes that everyone is a member of exactly one group in G. Thus, when viewed within the extended model, our model corresponds to the restricted domain $\mathfrak{B}^*$ consisting of all problems $\mathbf{B} \in \mathfrak{B}$ such that $\sum_{k\in G} \mathbf{B}^k = 1_{n\times n}$. Then for all $\mathbf{B} \in \mathfrak{B}^*$ and all distinct pairs $k, k' \in G$, $\mathbf{B}^{k\wedge k'} = 0_{n\times n}$ and $\mathbf{B}^{1\vee\cdots\vee m} = 1_{n\times n}$. Note that on this restricted domain, all persons agree with partitioning N into basic groups and so *basic group partitioning* is equivalent to *unanimous basic group partitioning*.

Now we establish a result that corresponds to Theorem 1. Due to *independence of irrelevant opinions*, our result holds both on the restricted domain $\mathfrak{B}^*$ and the whole domain $\mathfrak{B}$.

Proposition 1. (i) *If an extended rule F on $\mathfrak{B}^*$ (or $\mathfrak{B}$) satisfies* independence of irrelevant opinions, basic group non-degeneracy, *and* unanimous basic group partitioning, *then there is an extended one-vote rule $\hat{F}$ such that for all $\mathbf{B} \in \mathfrak{B}^*$ (or $\mathfrak{B}$) and all $k \in G$, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.*
(ii) *An extended rule F on $\mathfrak{B}^*$ (or $\mathfrak{B}$) satisfies* independence of irrelevant opinions, basic group non-degeneracy, unanimous basic group partitioning, *and* consistency *if and only if it is an extended one-vote rule.*

Proof. We prove part (i) for the domain $\mathfrak{B}$ and skip the simpler proof for $\mathfrak{B}^*$. Let F be an extended rule on $\mathfrak{B}$ satisfying the three axioms. Let f be the rule in our model defined as follows using F: for all $P \in \mathcal{P}$, all $k \in G$, and all $i \in N$,

$$f_i(P) = k \Leftrightarrow F_i^k(\mathbf{B}) = 1 \ , \tag{1}$$

where $\mathbf{B}$ is such that for all $\ell \in G$, $\mathbf{B}^\ell = B^{P,\ell}$. By *unanimous basic group partitioning*, f is well-defined. *Independence of irrelevant opinions* and *basic group non-degeneracy* of F imply *independence of irrelevant opinions* and *non-degeneracy* of f, respectively. Thus by Theorem 1, f is a one-vote rule; there is $h\colon N \to N \times N$ such that for all $P \in \mathcal{P}$, $f_i(P) = P_{h(i)}$. Let $\hat{F}$ be the extended one-vote rule such that for all $\mathbf{B} \in \mathfrak{B}$, all $g \in \mathcal{G}\backslash\{\mathfrak{1}, \mathfrak{o}\}$, and all $i \in N$, $\hat{F}_i^g(\mathbf{B}) = \mathbf{B}_{h(i)}^g$.

Let $\mathbf{B} \in \mathfrak{B}^*$. There is $P \in \mathcal{P}$ such that for all $k \in G$, $\mathbf{B}^k = B^{P,k}$. Since $f_i(P) = P_{h(i)}$, (1) implies that for all $k \in G$, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.

Let $\mathbf{B} \in \mathfrak{B}\backslash\mathfrak{B}^*$. Let $k \in G$. There are binary problems $(\mathbf{B}'^\ell)_{\ell \in G\backslash\{k\}}$ such that $\mathbf{B}^k + \sum_{\ell \in G\backslash\{k\}} \mathbf{B}'^\ell = 1_{n\times n}$. Then the profile $\big(\mathbf{B}^k, (\mathbf{B}'^\ell)_{\ell \in G\backslash\{k\}}\big)$ of binary problems for all basic groups generates an extended problem in $\mathfrak{B}^*$. Denote by $\mathbf{B}''$ the extended problem so generated. By the argument in the previous paragraph, $F^k(\mathbf{B}'') = \hat{F}^k(\mathbf{B}'')$. Note that $\mathbf{B}''^k = \mathbf{B}^k$. Since both F and $\hat{F}$ are *independent of irrelevant opinions*, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.

Part (ii) follows directly from part (i) and *consistency*. □

Remark 1. Part (i) shows that even without *consistency*, we cannot get much far away from the extended one-vote rules. It also reveals that the family of rules we characterize in the multinary model is similar to but larger than the family of rules Miller (2008) characterizes.

On the other hand, if *independence of irrelevant opinions* is dropped in part (ii), one can find a rich family of rules, quite different from the one-vote rules, satisfying the other axioms. For example, fix a default group $\nu \in G$ and a profile of consent quotas $q \equiv (q_k)_{k \in G \backslash \{\nu\}}$ for basic groups other than ν, where for all $k \in G \backslash \{\nu\}$, $q_k \in \{1, \cdots, n\}$. Define an extended rule ${}^{q,\nu}F$ as follows: for all $\mathbf{B} \in \mathfrak{B}$ and all $i \in N$, (i) for all $k \in G \backslash \{\nu\}$, (i.a) if $\mathbf{B}^k_{ii} = 1$ and $|\{j \in N : \mathbf{B}^k_{ji} = 1\}| \geq q_k$, then ${}^{q,\nu}F^k_i(\mathbf{B}) = 1$, (i.b) otherwise, ${}^{q,\nu}F^k_i(\mathbf{B}) = 0$; (ii) ${}^{q,\nu}F^\nu_i(\mathbf{B}) = 1$ if and only if for all $k \in G \backslash \{\nu\}$, ${}^{q,\nu}F^k_i(\mathbf{B}) = 0$; and (iii) for all derived groups $g \in \mathcal{G} \backslash G$, ${}^{q,\nu}F^g_i(\mathbf{B})$ is the consistent extension of the basic group decisions $({}^{q,\nu}F^k(\mathbf{B}))_{k \in G}$. These rules satisfy all axioms in in the proposition except for *independence of irrelevant opinions*. To take another example, consider the *extended plurality rule*, denoted PL (similar to the plurality rules in our model) and defined as follows: for all $\mathbf{B} \in \mathfrak{B}$, (i) for all $k \in G$, and all $i \in N$, $PL^k_i(\mathbf{B}) = 1$ if and only if for all $k' \in G$, $|\{j \in N : \mathbf{B}^k_{ji} = 1\}| \geq |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|$; (ii) for all other derived groups $g \in \mathcal{G} \backslash G$, $PL^g(\mathbf{B})$ is the consistent extension of the basic group decisions $\left(PL^k(\mathbf{B})\right)_{k \in G}$. It is clear that this rule violates *independence of irrelevant alternatives* but satisfies *non-degeneracy*. However, there can be two or more basic groups to which a person i belongs under PL. This can occur even for problems in $\mathfrak{B}^*$ and so PL violates *unanimous basic group partitioning*. However, we can define a "refinement" of PL using a linear ordering $\succ$ over basic groups, denoted ${}^{\succ}PL$, as follows: for all $\mathbf{B} \in \mathfrak{B}$, (i) for all $k \in G$, and all $i \in N$, ${}^{\succ}PL^k_i(\mathbf{B}) = 1$ if and only if for all $k' \in G \backslash \{k\}$, either $[|\{j \in N : \mathbf{B}^k_{ji} = 1\}| > |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|]$ or $[|\{j \in N : \mathbf{B}^k_{ji} = 1\}| = |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|$ and $k' \succ k]$; (ii) for all derived groups $g \in \mathcal{G} \backslash G$, ${}^{\succ}PL^g(\mathbf{B})$ is the consistent extension of the basic group decisions $\left({}^{\succ}PL^k(\mathbf{B})\right)_{k \in G}$. Then this rule satisfies *unanimous basic group partitioning* too. We, therefore, find that in part (ii) of the proposition, *independence of irrelevant opinions* plays a much more significant role than *consistency*.

Also, some extended one-vote rules violate *basic group partitioning* on $\mathfrak{B}$. Thus, *unanimous basic group partitioning* in the proposition cannot be replaced by *basic group partitioning*. △

In Miller's (2008) model, a decision rule takes as an argument only one component of $\mathbf{B}$, say $\mathbf{B}^g$ for some $g \in \mathcal{G}$. Thus, the identification of group g only depends on $\mathbf{B}^g$. This means that the following property is implicitly assumed.

Component-wise Independence. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $g \in \mathcal{G}$, if $\mathbf{B}^g = \mathbf{B}'^g$, then $F^g(\mathbf{B}) = F^g(\mathbf{B}')$.

This axiom requires independent decision making across derived groups as well as across basic groups. It is quite stronger than *independence of irrelevant opinions*. In fact, most extended consent rules violate it. For example, consider the extended "majority" consent rule with $s = t = \frac{n+1}{2}$. Then for all $i \in N$ and all $k \in G$, person i belongs to group k if and only if a majority believes her to be a member of group k, that is, $|\{j \in N \colon \mathbf{B}^k_{ji} = 1\}| \geq \frac{n+1}{2}$. Let $i \in N$. Consider $\mathbf{B} \in \mathfrak{B}$ such that (i) a majority identifies i as a member of group k; (ii) a majority identifies i as a member of group ℓ, and (iii) only a minority (less than $\frac{n+1}{2}$) identifies i as a member of group $k \wedge \ell$. Then for $\mathbf{B}$, the extended majority consent rule determines i as a member of groups k (by (i)), ℓ (by (ii)), and $k \wedge \ell$ (by *consistency*). Now consider $\mathbf{B}' \in \mathfrak{B}$ such that (i′) a minority identifies i as a member of group k; and (ii′) $\mathbf{B}'^{k\wedge\ell} = \mathbf{B}^{k\wedge\ell}$. For $\mathbf{B}'$, the extended majority consent rule determines i as a member of neither k (by (i′)) nor $k \wedge \ell$ (by *consistency*). Thus, although $\mathbf{B}$ and $\mathbf{B}'$ agree on the membership for group $k \wedge \ell$, the extended majority consent rule assigns different decisions to them, violating *component-wise independence*. It can be shown that only one extended consent rule is *component-wise independent*: the extended consent rule with $s = t = 1$, namely the liberal rule.

Adding *component-wise independence*, the characterization of the extended one-vote rules as in Miller (2008) can be obtained directly from his result.

Proposition 2 (Miller, 2008)**.** *An extended rule on* $\mathfrak{B}$ *satisfies* component-wise independence*,* non-degeneracy*, and* (meet *and* join) consistency *if and only if it is an extended one-vote rule.*

Proof. To prove the nontrivial direction, let F be an extended rule satisfying the three axioms. Define $\phi \colon \mathcal{G} \times \mathcal{B} \to \{0,1\}^N$ such that for all $g \in \mathcal{G}$ and all $B \in \mathcal{B}$, $\phi(g, B) \equiv$

$F^g(\mathbf{B})$ for some $\mathbf{B} \in \mathfrak{B}$ with $\mathbf{B}^g = B$. Then by By *component-wise independence*, ϕ is well defined and for all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, $F^g(\mathbf{B}) = \phi(g, \mathbf{B}^g)$. Now applying Theorem 2.5 in Miller (2008) to $\phi(\cdot)$, we conclude that F is an extended one-vote rule. □

In fact, we can obtain results that are stronger than Proposition 2 as corollaries to Theorem 1. Further, it turns out that the "decisive votes" that feature the extended one-vote rules emerge even in the absence of *consistency*. To this end, we introduce three axioms whose scope of application is restricted to complete groups. The following axiom, weaker than *component-wise independence*, requires decisions to be independent across complete groups.

Complete Group Independence. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $g \in \mathcal{G}^c$, if $\mathbf{B}^g = \mathbf{B}'^g$, then $F^g(\mathbf{B}) = F^g(\mathbf{B}')$.

Next is the restriction of *non-degeneracy* to complete groups.

Complete Group Non-Degeneracy. For all $g \in \mathcal{G}^c$ and all $i \in N$, there exist $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^g(\mathbf{B})$=0 and $F_i^g(\mathbf{B}') = 1$.

Finally, since complete groups are mutually exclusive, we require that the decisions on complete groups partition N. This is weaker than *consistency*.

Complete Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, $\left(\{i \in N : F_i^g(\mathbf{B}) = 1\}\right)_{g \in \mathcal{G}^c}$ is a partition of N.

Applying Theorem 1, we obtain the following result.

Proposition 3. (i) *If an extended rule F on $\mathfrak{B}$ satisfies* complete group independence*,* complete group non-degeneracy*, and* complete group partitioning*, then there is an extended one-vote rule $\hat{F}$ such that for all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}^c$, $F^g(\mathbf{B}) = \hat{F}^g(\mathbf{B})$.*
(ii) *An extended rule F on $\mathfrak{B}$ satisfies* complete group independence*,* complete group non-degeneracy*, and* consistency *if and only if it is an extended one-vote rule.*

Proof. Part (i). Let F be a rule satisfying the three axioms. Let $K \equiv \{1, \cdots, 2^m\}$ and fix a one-to-one correspondence $\theta\colon K \to \mathcal{G}^c$. For each $\mathbf{B} \in \mathfrak{B}$, by *opinion-consistency*, there is $P \in K^{N \times N}$ such that for all $i, j \in N$ and all $k \in K$,

$$P_{ij} = k \Leftrightarrow \mathbf{B}_{ij}^{\theta(k)} = 1\ . \tag{2}$$

Conversely, for each $P \in K^{N \times N}$, there is $\mathbf{B} \in \mathfrak{B}$ satisfying (2). In fact, (2) defines a one-to-one correspondence $\Theta\colon K^{N \times N} \to \mathfrak{B}$. Let $f\colon K^{N \times N} \to K^N$ be defined as follows: for all $P \in K^{N \times N}$ and all $i \in N$, $f_i(P) = k$ if $F_i^{\theta(k)}(\Theta(P)) = 1$. By *complete group partitioning*, f is well-defined. *Complete group independence* of F implies *independence of irrelevant opinions* of f. *Complete group non-degeneracy* of F implies *non-degeneracy* of f. Finally, applying Theorem 1, we conclude that f is a one-vote rule, which gives the desired conclusion.

Part (ii). To prove the non-trivial direction, let F be a rule satisfying *complete group independence*, *complete group non-degeneracy*, and *consistency*. Since *consistency* implies *complete group partitioning*, part (i) holds and the decision on any complete group by F coincides with the decision by an extended one-vote rule. Then by *consistency*, we conclude that the decision on any other group by F also coincides with the decision by the extended one-vote rule. □

Remark 2. This proposition can be used to prove Theorem 2.5 in Miller (2008) as follows. Consider a rule $\phi\colon \mathcal{G} \times \mathcal{B} \to \{0,1\}^N$ satisfying the axioms of *consistency* ("separability" as Miller calls it) and *non-degeneracy* in his theorem. Now use this rule ϕ to define an extended rule F such that for all $\mathbf{B} \in \mathfrak{B}$, all $g \in \mathcal{G}$, and all $i \in N$, $F_i^g(\mathbf{B}) \equiv \phi_i(g, \mathbf{B}^g)$. By construction, F satisfies *component-wise independence*. Also, *consistency* and *non-degeneracy* of ϕ imply *consistency* and *non-degeneracy* of F, respectively. Hence, part (ii) of the proposition implies that F is an extended one-vote rule and ϕ is a one-vote rule, as defined in Miller's (2008) model. △

Remark 3. Part (ii) of the proposition strengthens Proposition 2 by weakening *component-wise independence* to *complete group independence*. Also, part (i) indicates that even if *consistency* is dropped in part (ii) (while *complete group partitioning* is retained), we cannot get that far way from the extended one-vote rules. Any rule satisfying the other axioms coincides with an extended one-vote rule on its decisions for complete groups. However, if *complete group independence* is dropped, all extended consent rules satisfy the other axioms in part (ii). As in Remark 1, the independence axiom here, *complete group independence,* plays a much more significant role than *consistency*. △

Note that on the restricted domain $\mathfrak{B}^*$, *independence of irrelevant opinions* is equivalent to *complete group independence*. Also, in the presence of the basic property "unanimity" (if $\mathbf{B}^k = 0_{n\times n}$, $F^k(\mathbf{B}) = 0_{1\times n}$), (*unanimous*) *basic group partitioning* is equivalent to *complete group partitioning*. Thus, on the restricted domain $\mathfrak{B}^*$, we obtain almost the same results using the two sets of axioms.

It should be noted that both *component-wise independence* and *complete group independence* are quite more demanding than *independence of irrelevant opinions*. For simplicity, suppose that $G = \{a, b\}$. *Component-wise independence* and *complete group independence* require that for a decision on who are "a and b" ($a \wedge b$), we should only pay attention to opinions on who are "a and b" ($a \wedge b$) and ignore any other opinions, including the opinions on who are a and on who are b, which do not seem irrelevant to the decision on "a and b". For example, there are many cases of $\mathbf{B}^a$ and $\mathbf{B}^b$ which give rise to the same opinions on $a \wedge b$, $\mathbf{B}^{a\wedge b} = 0_{n\times n}$, such as when $\mathbf{B}^a = 0_{n\times n}$ and $\mathbf{B}^b = 0_{n\times n}$, when $\mathbf{B}^a = 1_{n\times n}$ and $\mathbf{B}^b = 0_{n\times n}$, and when $\mathbf{B}^a = \begin{pmatrix} 1_{k\times k} & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & 1_{(n-k)\times(n-k)} \end{pmatrix}$ and $\mathbf{B}^b = \begin{pmatrix} 0_{k\times k} & 1_{k\times(n-k)} \\ 1_{(n-k)\times k} & 0_{(n-k)\times(n-k)} \end{pmatrix}$. The two independence axioms require that in all these different cases, there should be the same decision on $a \wedge b$, which seems too strong. Such an independence is not demanded by *independence of irrelevant opinions* since it requires independent decisions only across basic groups.